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MHD flow of a power-law fluid over a rotating disk ☆

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Abstract

Magnetohydrodynamic flow of an electrically conducting power-law fluid in the vicinity of a constantly rotating infinite disk in the presence of a uniform magnetic field is considered. The steady, laminar and axi-symmetric flow is driven solely by the rotating disk, and the incompressible fluid obeys the inelastic Ostwald de Waele power-law model. The three-dimensional boundary layer equations transform exactly into a set of ordinary differential equations in a generalized similarity variable. These ODEs are solved numerically for values of the magnetic parameter m up to 4.0. The effect of the magnetic field is to reduce, and eventually suppress, the radially directed outflow. An accompanying reduction of the axial flow towards the disk is observed, together with a thinning of the boundary layer adjacent to the disk, thereby increasing the torque required to maintain rotation of the disk at the prescribed angular velocity. The influence of the magnetic field is more pronounced for shear-thinning than for shear-thickening fluids.

Keywords: Non-Newtonian fluids; Von Karman swirling flow; Similarity solutions; Magnetohydrodynamics

1. Introduction

A seemingly simple type of genuine three-componential viscous flow is that induced by a disk rotating at a constant rate in an infinite fluid body which would otherwise be at rest. Since von Kármán [10] elegantly devised a similarity transformation, which reduced the full Navier–Stokes system to a set of ordinary differential equations in a single variable, this swirling flow has received considerable interest; see, e.g., the review by Zandbergen and Dijkstra [11].

Several variants of the problem considered by von Kármán have been studied over the years, of which only two have relevance for the present investigation. Sparrow and Cess [9] examined how the swirling flow caused by a rotating disk is affected by an axial magnetic field, whereas Mitschka [6] and Mitschka and Ulbrecht [7] extended von Kármán's analysis to non-Newtonian power-law fluids.

Motion of power-law fluids in the presence of a magnetic field has been studied earlier by several authors, e.g., [2,4,5,8]. Examples of non-Newtonian fluids which might be conductors of electricity were given by Sarpkaya [8], e.g., flow of nuclear slurries and of mercury amalgams, and lubrication with heavy oils and greases. The magnetohydrodynamic flow of a power-law fluid over a rotating disk has not been dealt with so far. This flow is of particular interest since the magnetic force field in this case vanishes outside the viscous boundary layer and therefore affects the fluid motion only within the boundary layer. The objective of the present paper is therefore to explore the influence of a magnetic field on the flow of a power-law fluid due to a rotating disk, and in particular how effectively a uniform magnetic field can be utilized as a means of flow control.

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2. Physical problem and mathematical formulation

The steady flow induced by a steadily rotating disk immersed in an otherwise quiescent electrically conducting fluid is considered. The infinite disk is rotating with constant angular velocity Ω about the z-axis, i.e. $r = 0$ in a cylindrical coordinate system. The conducting fluid is permeated by an imposed uniform magnetic field $\boldsymbol{B}$, which acts in the positive z-direction normal to the disk. The magnetic body force vector

$$\boldsymbol{f} = \sigma(\boldsymbol{E} + \boldsymbol{V} \times \boldsymbol{B}) \times \boldsymbol{B} \tag{1}$$

depends not only on $\boldsymbol{B}$, but also on the electric field vector $\boldsymbol{E}$ and the fluid velocity $\boldsymbol{V}$, whereas σ denotes the electrical conductivity of the fluid. In the low-magnetic-Reynolds-number approximation, in which the induced magnetic field can be ignored and the imposed and induced electric fields are assumed negligible, the force vector $\boldsymbol{f}$ simplifies to $f_r = -\sigma u B_0^2$, $f_\theta = -\sigma v B_0^2$, $f_z = 0$, where u, v and w denote the components of the velocity vector $\boldsymbol{V}$ in the radial, azimuthal and axial directions, respectively. The non-Newtonian fluid under consideration is assumed to obey the Ostwald-de Waele power-law model:

$$\tau = 2\mu D = 2K(2D_{ij}D_{ij})^{(n-1)/2}D, \tag{2}$$

where D denotes the deformation, or rate-of-strain, tensor and K and n are the consistency coefficient and the power-law index, respectively.

The motion of the inelastic power-law fluid, which is confined to the half-space $z > 0$ above the infinite disk, is governed by the boundary layer equations for conservation of mass and momentum:

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0, \tag{3}$$

$$\rho\left(u\frac{\partial u}{\partial r} - \frac{v^2}{r} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial r} + \frac{\partial}{\partial z}\left(\mu\frac{\partial u}{\partial z}\right) - \sigma u B_0^2, \tag{4}$$

$$\rho\left(u\frac{\partial v}{\partial r} + \frac{uv}{r} + w\frac{\partial v}{\partial z}\right) = \frac{\partial}{\partial z}\left(\mu\frac{\partial v}{\partial z}\right) - \sigma v B_0^2, \tag{5}$$

$$0 = -\frac{\partial p}{\partial z} \tag{6}$$

together with the boundary conditions

$$u = w = 0 \quad \text{and} \quad v = \Omega r \quad \text{at } z = 0 \quad \text{and} \tag{7}$$

$$u = v = 0 \quad \text{as } z \to \infty. \tag{8}$$

Here,

$$\mu = K\left\{\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2\right\}^{(n-1)/2} \tag{9}$$

is the viscosity function, which simplifies to $\mu = K$ for the particular parameter value $n = 1$, i.e. for Newtonian fluids.

3. Similarity transformation

The partial differential equations (3)–(5) reduce to those considered by Mitschka [6] and Andersson et al. [3] in the non-magnetic case $B_0 = 0$. Let us therefore introduce the generalized dimensionless similarity variable η originally proposed by Mitschka [6]:

$$\eta = z\left(\frac{\Omega^{2-n}}{K/\rho}\right)^{1/(n+1)} \cdot r^{(1-n)/(1+n)} \tag{10}$$

together with a von Kármán type representation of the velocity field:

$$u = r\Omega \cdot F(\eta), \qquad v = r\Omega \cdot G(\eta), \qquad w = \left(\frac{\Omega^{1-2n}}{K/\rho}\right)^{-1/(n+1)} \cdot r^{(n-1)/(n+1)} \cdot H(\eta). \tag{11}$$

In accordance with von Kármán's [10] original similarity transformation, the pressure p is assumed to be a function of z only. The obvious implication there of is that $\partial p/\partial r = 0$ in Eq. (4) and the reduced momentum balance (6) in the axial direction gives that p remains constant across the boundary layer. Accordingly, the pressure is everywhere constant and thus equal to the ambient pressure.

In spite of the introduction of the magnetic force field in the present study, the boundary layer Eqs. (3)–(5) for mass continuity, radial and azimuthal momentum transform into the ordinary differential equations:

$$H' = -2F - \frac{1-n}{n+1}\eta F', \tag{12}$$

$$F^2 - G^2 + \left(H + \frac{1-n}{n+1}F\eta\right)F' + mF = \left[\left(F'^2 + G'^2\right)^{(n-1)/2} \cdot F'\right]', \tag{13}$$

$$2FG + \left(H + \frac{1-n}{n+1}F\eta\right)G' + mG = \left[\left(F'^2 + G'^2\right)^{(n-1)/2} \cdot G'\right]', \tag{14}$$

where the primes denote differentiation with respect to the similarity variable η and m is the magnetic parameter

$$m = \sigma B_0^2/\rho\Omega. \tag{15}$$

The boundary conditions (7) and (8) transform into

$$F = 0, \qquad G = 1, \qquad H = 0 \qquad \text{at } \eta = 0, \tag{16}$$

$$F = 0, \qquad G = 0 \qquad \text{as } \eta \to \infty, \tag{17}$$

respectively.

Of particular practical relevance is the momentum coefficient:

$$C_M = \frac{-2M}{(1/2)\rho\Omega^2 r_0^5} = -8\pi\left(\frac{n+1}{5n+3}\right)\mathrm{Re}_0^{-1/(n+1)}\left[F'(0)^2 + G'(0)^2\right]^{(n-1)/2} \cdot G'(0), \quad \text{where} \tag{18}$$

$$M = \int_0^{r_0} r\tau_{\theta z} 2\pi r \, \mathrm{d}r \tag{19}$$

is the torque required to maintain steady rotation of a disk with radius r_0 and

$$\mathrm{Re}_0 = \rho(r_0\Omega)^{2-n} r_0^n/K \tag{20}$$

is a local Reynolds number based on r_0.

The ordinary differential equations (12)–(14) together with the boundary conditions (16) and (17) constitute a two-parameter two-point boundary value problem which can be solved numerically by the same technique as that described by Andersson et al. [3]. First, the ODEs were written as a system of five coupled first-order equations and thereafter all derivatives were replaced by finite-difference approximations.

4. Asymptotic solution for large *m*

It can be conjectured that the negative body force, which results from the imposition of a magnetic field, tends to reduce the radial (F) and azimuthal (G) velocity components, and, therefore, in turn also reduces the axial velocity (H). For sufficiently high values of the magnetic parameter m, both F and H may vanish, whereas G inevitably remains finite in the immediate vicinity of the disk due to the no-slip constraint $G = 1$. Therefore, in the limit $m \to \infty$, the azimuthal momentum Eq. (14) reduces to:

$$\left[\left(G'^2\right)^{(n-1)/2} \cdot G'\right]' = mG. \tag{21}$$

Let us furthermore assume that $G' \leqslant 0$. Then, with $G''\,\mathrm{d}G = G'\,\mathrm{d}G'$, this becomes

$$-n(-G')^n \,\mathrm{d}G' = mG\,\mathrm{d}G \tag{22}$$

which can be integrated once to give

$$-G' = \left(\frac{n+1}{2n} mG^2\right)^{1/(n+1)}, \tag{23}$$

where it has been assumed that both G and G' tend to zero sufficiently far away from the disk. The intermediate result (23) can be integrated once more to give the closed-form solution

$$G(\eta) = \exp\left(-m^{1/2}\eta\right) \quad \text{for } n = 1, \tag{24a}$$

$$G(\eta) = \left\{1 + \frac{1-n}{1+n}\left(\frac{1+n}{2n}m\right)^{1/(1+n)} \cdot \eta\right\}^{-(1+n)/(1-n)} \quad \text{for } n \neq 1 \tag{24b}$$

which satisfies the boundary condition $G(0) = 1$. Here, Eq. (24a) is the same asymptotic solution as that obtained by Sparrow and Cess [9] for a Newtonian fluid, whereas Eq. (24b) applies for both shear-thinning ($n < 1$) and shear-thickening ($n > 1$) fluids. Of particular practical relevance is the wall-gradient

$$-G'(0) = \left(\frac{n+1}{2n}m\right)^{1/(n+1)} \tag{25}$$

which determines the torque required to maintain the rotation with constant angular velocity; cf. Section 5.3 for further details.

5. Results

5.1. *Non-magnetic case $m = 0$*

Numerical results for the non-magnetic case $m = 0$ have recently been reported by Andersson et al. [3] for power-law fluids with n in the range from 0.2 to 2.0. These results are generally consistent with the earlier findings of Mitschka and Ulbrecht [7] and show that the shear-driven motion (G) in the azimuthal direction decays rapidly with the distance η from the disk. The centrifugal force associated with this circular motion causes an outward radial flow (F), which in turn is compensated by an axial inflow ($-H$) towards the rotating disk. It is readily observed that the characteristic peak in the radial velocity distribution F is only modestly affected by variations in the power-law index and the only striking effect of the rheology is the monotonic thinning of the boundary layer with increasing n. The same thinning effect with increasing shear-thickening has also been found in two-dimensional boundary layer flows of power-law fluids; see, e.g., [1]. An ambiguity in the determination of the axial inflow $-H(\infty)$ towards the disk for highly shear-thinning fluids was revealed by Andersson et al. [3] and ascribed to a breakdown of the boundary layer approximation.

5.2. *Sample case $n = 0.5$*

The effect of the imposition of a magnetic field on the flow of a moderately shear-thinning or pseudo-plastic fluid with $n = 0.5$ is shown in Figs. 1–3. The negative magnetic body force in the transformed momentum equations (13) and (14) effectively reduces the radial and azimuthal velocity components with increasing m-values. The accompanying reduction of the axial flow (H) in Fig. 3, however, is an indirect effect caused by mass continuity, since the axial inflow is aimed to compensate the radial outflow. The same effect of the magnetic force field is observed for all values of the power-law index in the range $0.2 < n < 2.0$, including the special case $n = 1$ (i.e. Newtonian fluids) which was considered by Sparrow and Cess [9]. It is worthwhile to mention that the present results for $n = 1$ compare to within 0.1 per cent with those reported by Sparrow and Cess [9], as can be seen from Table 1.

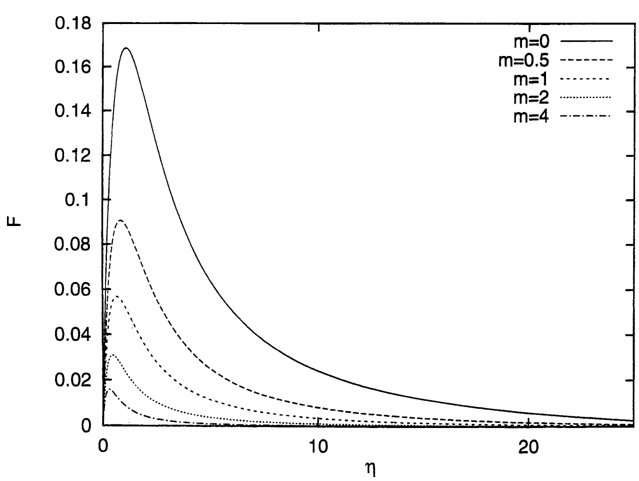

Fig. 1. Radial velocity component F for a shear-thinning fluid with $n = 0.5$ for different values of the magnetic parameter m.

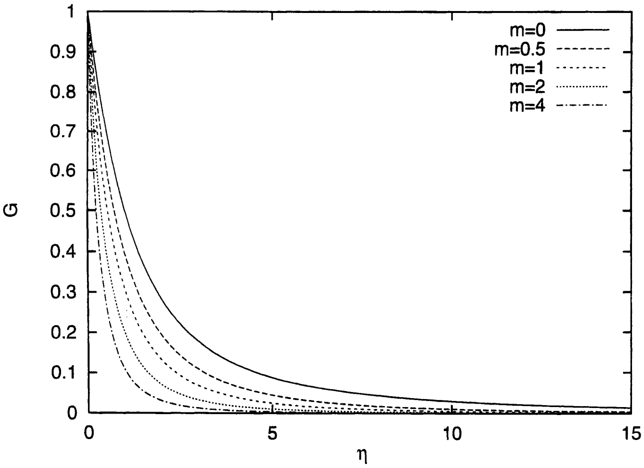

Fig. 2. Azimuthal velocity component G for a shear-thinning fluid with $n = 0.5$ for different values of the magnetic parameter m.

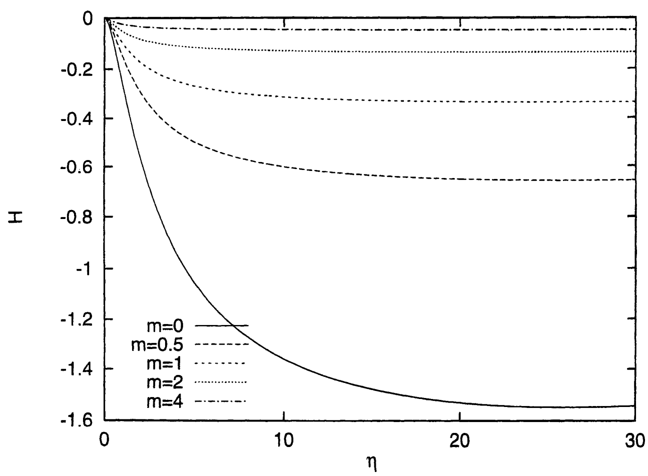

Fig. 3. Axial velocity component H for a shear-thinning fluid with $n = 0.5$ for different values of the magnetic parameter m.

Table 1
Comparison of some flow characteristics with results of Sparrow and Cess [9] (S & C) for the particular case of a Newtonian fluid ($n = 1$)

Magnetic parameter m	$F'(0)$		$-G'(0)$		$-H(\infty)$	
	Present	S & C*	Present	S & C	Present	S & C
0	0.5101	0.5105	0.6160	0.616	0.8827	0.885
0.5	0.3851	0.3850	0.8487	0.849	0.4589	0.459
1	0.3093	0.3095	1.0691	1.069	0.2533	0.253
2	0.2306	0.2305	1.4421	1.442	0.1086	0.109
4	0.1657	0.1655	2.0103	2.010	0.0408	0.0408

* The entries in this column are taken from [9, Table 1] and divided by two.

5.3. Parametrization

Since the same overall effects of the magnetic field as described above for $n = 0.5$ are observed for all power-law fluids, shear-thickening as well as shear-thinning, only the primary flow characteristics $F'(0)$, $G'(0)$, and $H(\infty)$ are reported for the whole parameter range of n in Figs. 4–6. Because the dimensionless azimuthal velocity $G(\eta)$ inevitably has to change from 1 at the disk to zero at infinity, the retardation of the azimuthal velocity caused by the magnetic force field implies an increase in the

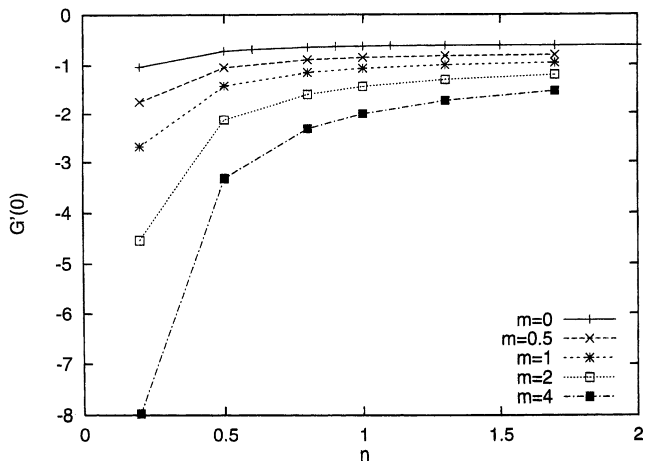

Fig. 4. Variation of the wall-gradient $G'(0)$ with power-law index n for different values of the magnetic parameter m.

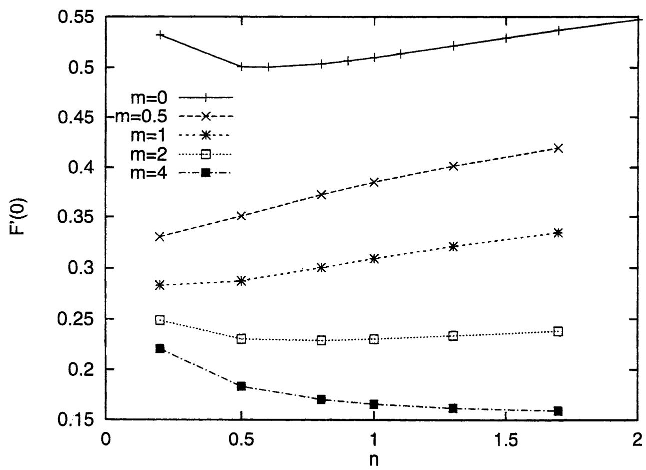

Fig. 5. Variation of the wall-gradient $F'(0)$ with power-law index n for different values of the magnetic parameter m.

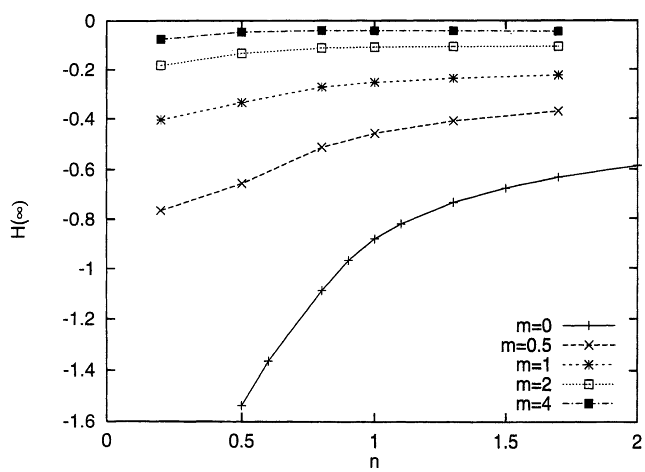

Fig. 6. Variation of the asymptotic axial flow $H(\infty)$ with power-law index n for different values of the magnetic parameter m.

magnitude of the accompanying velocity gradient G' at the disk, as shown in Fig. 4. The substantial effect of the magnetic field component $f_\theta = -\sigma v B_0^2$ can easily be understood since v attains its highest values in the immediate vicinity of the rotating disk and decays to zero far away from the disk surface, cf. Fig. 2. Since the boundary layer thickness, or equivalently, the layer with non-zero azimuthal velocity, decreases with increasing n-values, it is not at all surprising to observe from Fig. 4 that the effect of the magnetic field is more pronounced for shear-thinning than for shear-thickening fluids.

While the wall-gradient of the radial velocity profile decreases monotonically with m due to the retarding magnetic force field component, as shown in Fig. 5, the variation of $F'(0)$ with the power-law index is more subtle. In the non-magnetic case, $F'(0)$ attains a minimum level near $n = 0.5$, in accordance with the earlier findings of Mitschka and Ulbrecht [7] and confirmed by Andersson et al. [3]. It has moreover been observed that the peak in the F-profile shifts inwards with increasing n-values, thereby explaining the increasing $F'(0)$ with n for $n > 0.5$. The imposition of the magnetic field further enhances the inward shift of the peak of $F(\eta)$, but the effect of shifting the peak position is more than outweighed by the substantial reduction of the peak level for the strong magnetic fields, i.e. $m > 1.0$. This explains why $F'(0)$ increases with n for modest magnetic fields $m < 1.0$, but decreases with increasing shear-thickening for $m > 1.0$. It is worthwhile to recall that for non-Newtonian fluids both $F'(0)$ and $G'(0)$ contribute to the momentum coefficient defined in Eq. (18). According to the results presented in Figs. 4 and 5, C_M increases with m for all values of n considered here.

The axial inflow $-H(\infty)$ in Fig. 6 shows a simple dependence on the magnetic parameter m and the power-law index n. Since the level of $F(\eta)$ is reduced with increasing magnetic fields for all values of of n, the axial inflow $-H(\infty)$ is also reduced in accordance with

$$-H(\infty) = 2\int_0^\infty F\,\mathrm{d}\eta + \frac{1-n}{n+1}\int_0^\infty F'\eta\,\mathrm{d}\eta \tag{26}$$

deduced from Eq. (12) by Andersson et al. [3]. The effect of the magnetic field is more pronounced for shear-thinning fluids than for shear-thickening fluids, obviously because the boundary layer thickness, and thus the impact of the magnetic force, is decreasing monotonically with n. These observations have direct implications on the pumping efficiency of the disk, of which the flow rate

$$Q = \int_0^{r_0} -w(\infty)2\pi r\,\mathrm{d}r = -H(\infty)\frac{2(n+1)}{3n+1}\pi r_0^2\big[\Omega^{2n-1}r_0^{n-1}K/\rho\big]^{1/(n+1)} \tag{27}$$

is the most appropriate measure.

6. Conclusions

The MHD flow problem was solved numerically for values of the magnetic parameter m up to 4.0. The effect of the magnetic field is to reduce, and eventually suppress, the radially directed outflow. An accompanying reduction of the axial flow towards the disk is observed, together with a thinning of the boundary layer adjacent to the disk. Moreover, due to the observed thinning of the boundary layer, the wall shear stress in the circumferential direction turns out to increase monotonically with increasing m-values, thereby increasing the torque required to maintain rotation of the disk at the prescribed angular velocity. The pumping efficiency Q is, on the other hand, substantially reduced with increasing the magnetic field. These major effects of the imposition of a magnetic field are found both for shear-thinning ($n < 1$) and shear-thickening ($n > 1$) fluids. The influence of the magnetic field is, however, more pronounced for shear-thinning than for shear-thickening fluids. The magnetic field therefore makes the difference between the various fluids, i.e. between different values of n, more distinct than in the non-magnetic case $m = 0$.

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